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A NOTE ON STRESS FUNCTIONS

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Abstract—Some general expressions for solenoidal stress fields are obtained from an application of the Stokes–Helmholtz decomposition as it relates to symmetric dyadics. Using these in conjunction with the Beltrami–Michell equations, the known solutions of the equations of classical elasticity are deduced.

INTRODUCTION

IF $\mathbf{S}$ is the stress dyadic† of an equilibrated body occupying the region Δ, then in the absence of body force

$$\nabla \cdot \mathbf{S} = 0 \tag{1}$$

in D, the interior of Δ. A solution of this equation in the form

$$\mathbf{S} = \nabla \times \mathbf{T} \times \nabla \tag{2}$$

where $\mathbf{T}$ is a symmetric dyadic, was presented by Beltrami [2]. If ∂D^* is any closed surface in Δ, $\mathbf{n}$ its outer normal and $\mathbf{r}$ the position vector, then the resultant traction on ∂D^* is

$$\int_{\partial D^*} \mathbf{n} \cdot \mathbf{S} \, dQ = \mathbf{t}^*, \qquad \int_{\partial D^*} \mathbf{r} \times (\mathbf{n} \cdot \mathbf{S}) \, dQ = \mathbf{m}^*. \tag{3}$$

When $\mathbf{S}$ is given by equation (2), $\mathbf{t}^* = \mathbf{m}^* = 0$ as a direct consequence of Stokes' theorem [3]. In other words, the resultant traction on every ∂D^*, arising from such stress fields, is zero. Keeping in mind that the reductions of these integrals, via the quoted theorem, impose some continuity restrictions on $\mathbf{T}$, we conclude, that when this is the case, equation (2) cannot represent all solenoidal dyadics. Gurtin [4] showed that if $\mathbf{S}$ meets equation (3) with $\mathbf{t}^* = \mathbf{m}^* = 0$ for every ∂D^* in Δ, then all sufficiently smooth solutions of equation (1) are represented by equation (2) provided ∂D, the boundary of D, is also smooth enough. Recently, Carlson [5] established the same result more directly. Gurtin also provided

$$\mathbf{S} = \nabla \times \mathbf{T} \times \nabla + \nabla^2(\nabla \mathbf{g} + \mathbf{g}\nabla) - \nabla(\nabla \cdot \mathbf{g})\nabla, \qquad \nabla^4 \mathbf{g} = 0 \tag{4}$$

as a representation for *all* smooth solutions of equation (1). His point of departure was the identity

$$\nabla^4 \mathbf{T} = \nabla \times \nabla \times \mathbf{T} \times \nabla \times \nabla + \nabla^2[\nabla(\nabla \cdot \mathbf{T}) + (\mathbf{T} \cdot \nabla)\nabla] - \nabla\nabla \cdot \mathbf{T} \cdot \nabla\nabla \tag{5}$$

By setting $\mathbf{S} = \nabla^4 \mathbf{T}$ and invoking equation (1), it follows that $\mathbf{g} = \nabla \cdot \mathbf{T}$ is biharmonic. Clearly, a Poisson type integral for biharmonic operators guarantees the existence of $\mathbf{T}$ for reasonable $\mathbf{S}$ and equation (4) follows. Subsequent unpublished work of a similar nature employing the second order identity

$$\nabla^2 \mathbf{U} = \nabla \times [\mathbf{U} - (\mathbf{U} : \mathbf{I})\mathbf{I}] \times \nabla + \nabla(\nabla \cdot \mathbf{U}) + (\mathbf{U} \cdot \nabla)\nabla - \mathbf{I}(\nabla \cdot \mathbf{U} \cdot \nabla), \qquad \mathbf{U} = \mathbf{U}_c \tag{6}$$

† Gibbs' notation is used throughout. For details, see [1].

led him to

$$\mathbf{S} = \nabla\times\mathbf{T}\times\nabla+\nabla\boldsymbol{\phi}+\boldsymbol{\phi}\nabla-\mathbf{I}(\nabla\cdot\boldsymbol{\phi}), \qquad \nabla^2\boldsymbol{\phi} = 0. \tag{7}$$

The purpose of this note is to obtain some general solutions of equation (1) through the use of the Stokes–Helmholtz decomposition for dyadics, to relate them to equations (4) and (5) and investigate the structure of stress functions in periphractic and multiply-connected regions. We further supply suitable forms for $\mathbf{T}$ whenever $\mathbf{S}$ also meets the Beltrami–Michell equations.

STOKES–HELMHOLTZ THEOREM

In order to express precise requirements, we first state a theorem on the continuity of the Newtonian potential relevant to this work which leads to smoothness restrictions in the S–H decomposition. In all that follows Δ is a bounded closed domain with interior D and whose boundary ∂D is sufficiently smooth.

THEOREM 1. *Suppose $f(P)$ is defined in Δ. If $f(P)$ has Hölder continuous m-th derivatives with exponent α in Δ, then the Newtonian potential*

$$g(P) = -(\tfrac{1}{4}\pi)\int_\Delta (f/R)\,\mathrm{d}Q \equiv \mathcal{N}[f]$$

has Hölder continuous $m+2$ derivatives with exponent α in Δ.

This theorem is a special case of the Holder–Korn–Lichtenstein–Giraud inequality [6]† and with it we can now state

THEOREM 2‡. *If $\mathbf{A}^{n+1}$ is an $(n+1)$-adic $(n = 0, 1, \ldots)$ which is $m+\alpha$ Hölder continuous in Δ, then there exists an n-adic $\mathbf{B}^n$ and an $(n+1)$-adic $\mathbf{C}^{n+1}$ that are $m+1+\alpha$ Hölder continuous in Δ, such that*

$$\mathbf{A}^{n+1} = \nabla\mathbf{B}^n+\nabla\times\mathbf{C}^{n+1} \tag{8}$$

in Δ.

This statement is a consequence of the identity $\nabla\times\nabla\times\mathcal{N} = \nabla\nabla\cdot\mathcal{N}-\nabla^2\mathcal{N}$ with $\mathbf{B}^n = \nabla\cdot\mathcal{N}$, $\mathbf{C}^{n+1} = -\nabla\times\mathcal{N}$ where $\mathcal{N}$ is the Newtonian potential whose density is $\mathbf{A}^{n+1}$. The usual phrasings of the Stokes–Helmholtz representation generally introduce Cauchy continuity hypotheses. As a result, they have the common defect that none provide the continuity of the function being represented that was assumed to establish the representation. (See, for example, [8, 9].)

THEOREM 3. *If $\mathbf{S}$ is $m+\alpha$ Hölder continuous in Δ and symmetric, then there exists a vector $\mathbf{s}$ and a dyadic $\mathbf{U}$, both $m+1+\alpha$ Hölder continuous in Δ such that*

$$\mathbf{S} = \nabla\mathbf{s}+\mathbf{s}\nabla+\nabla\times\mathbf{U}-\mathbf{U}_c\times\nabla \tag{9}$$

in Δ.

The theorem follows directly from Theorem 2 since in equation (8) we can choose $\mathbf{A}^{n+1}$ to be a dyadic $\mathbf{A}$ and observe its symmetric part can be assigned arbitrarily. Taking the expression for $\mathbf{A}$ and forming its conjugate, equation (9) is produced.

† Private remark by Lipman Bers, Columbia University.

‡ Dr. D. E. Carlson has informed me that this theorem along with the next two have already been presented by Mindlin [7].

By applying Theorem 2 to $\mathbf{U}_c$ we can write $\mathbf{U}_c = \nabla\mathbf{v}+\nabla\times\mathbf{W}$. Placing this into equation (9) there results

$$\mathbf{S} = \nabla(\mathbf{s}+\nabla\times\mathbf{v})+(\mathbf{s}+\nabla\times\mathbf{v})\nabla-\nabla\times(\mathbf{W}+\mathbf{W}_c)\times\nabla. \tag{10}$$

Tending to the continuity conditions of Theorem 2, we are led to

THEOREM 4. *If* $\mathbf{S}$ *is* $m+\alpha$ *Hölder continuous in* Δ *and symmetric, then there exists a vector* $\boldsymbol{\phi}$, $m+1+\alpha$ *and a symmetric dyadic* $\mathbf{T}$ $m+2+\alpha$ *Hölder continuous in* Δ *such that*

$$\mathbf{S} = \nabla\boldsymbol{\phi}+\boldsymbol{\phi}\nabla+\nabla\times\mathbf{T}\times\nabla \tag{11}$$

in Δ.

In closing this section, we recall that if $\mathbf{u}$ is the displacement field in a linear, homogeneous elastic equilibrated and isotropic body occupying the region Δ, then

$$(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}+\mu\nabla^2\mathbf{u} = 0 \tag{12}$$

in the absence of body force. λ, μ are Lamé's constants. If $\mathbf{u}$ is $m+\alpha$ $(m \geq 2)$ Hölder continuous in Δ, then the well-known Papkovich–Neuber solution [9] can be stated in the form

$$2\mu\mathbf{u} = \nabla(\phi+\mathbf{r}\cdot\boldsymbol{\psi})-4(1-\nu)\boldsymbol{\psi}, \qquad 2\nu(\lambda+\mu) = \lambda, \qquad \nu \neq \tfrac{1}{2} \tag{13}$$

where $\phi, \boldsymbol{\psi}$ are $m+1+\alpha$ Hölder continuous in Δ and regular harmonic functions in D.

By setting $\nabla^2\mathbf{v} = -4(1-\nu)\boldsymbol{\psi}$ and observing that $\nabla^2[\nabla\cdot\mathbf{v}+2(1-\nu)\mathbf{r}\cdot\boldsymbol{\psi}] = 0$, we find that $\nabla\cdot\mathbf{v}+2(1-\nu)\mathbf{r}\cdot\boldsymbol{\psi} = h$ where h is harmonic. Calling $2(1-\nu)\mathbf{w} = \nabla\mathcal{N}[h+2(1-\nu)\phi]$ and setting $\mathbf{g} = \mathbf{v}+[2(1-\nu)/(1-2\nu)]\mathbf{w}$ we obtain

$$2\mu\mathbf{u} = \nabla^2\mathbf{g}-\frac{1}{2(1-\nu)}\nabla\nabla\cdot\mathbf{g}, \qquad \nabla^4\mathbf{g} = 0 \tag{14}$$

where $\mathbf{g}$ is $m+2+\alpha$ Hölder continuous in Δ and is a regular biharmonic function in D. This is the Galerkin vector solution [10]. Equation (14) has already been obtained from equation (13) by Mindlin [11] by a different route.

Finally in equation (14) we set $\nabla^2\mathbf{g} = \mathbf{h}^*$. Clearly, $\nabla^2\mathbf{h}^* = 0$ and $\nabla\cdot\mathbf{g} = \mathcal{N}[\nabla\cdot\mathbf{h}^*]+h$ where h is harmonic. Placing these expressions into equation (14) we have

$$2\mu\mathbf{u} = \mathbf{h}^*-\frac{1}{2(1-\nu)}\nabla\{\mathcal{N}[\nabla\cdot\mathbf{h}^*]+h\}$$

which can be rewritten as

$$4\mu(1-\nu)\mathbf{u} = 2(1-\nu)\mathbf{h}^*-\nabla h-\nabla\mathcal{N}[\nabla\cdot\{2(1-\nu)\mathbf{h}^*-\nabla h\}]\frac{1}{2(1-\nu)}$$

Setting $\mathbf{h} = \mathbf{h}^*-\dfrac{1}{2(1-\nu)}\nabla h$ there results

$$2\mu\mathbf{u} = \mathbf{h}-[\tfrac{1}{2}(1-\nu)]\nabla\mathcal{N}[\nabla\cdot\mathbf{h}] \tag{15}$$

where $\mathbf{h}$ is $m+\alpha$ Hölder continuous in Δ and a regular harmonic vector D. Equation (15) is the Naghdi–Hsu solution [12].

SOLENOIDAL DYADICS

The representations of equations (9) and (11) in conjunction with equation (1) limit the functional character of the vectors $\mathbf{s}$ and $\boldsymbol{\phi}$. Beginning with equation (11), we find that $\mathbf{S}$

of this equation will be solenoidal provided

$$\nabla^2\boldsymbol{\phi}+\nabla\cdot(\nabla\boldsymbol{\phi}) \equiv \nabla\nabla\cdot\boldsymbol{\phi}+\nabla^2\boldsymbol{\phi} = 0. \tag{16}$$

But this is a special case of equation (12) for which $\mu = 1, \lambda = 0$, and these values of Lamé's constants imply $\nu = 0$. With these observations we can offer a number of representations for **S**. To begin we have

THEOREM 5. *If* **S** *is* $m+\alpha$ $(m \geq 1)$ *Hölder continuous in* Δ, *then*

$$\mathbf{S} = \nabla\boldsymbol{\phi}+\boldsymbol{\phi}\nabla+\nabla\times\mathbf{T}\times\nabla \tag{17}$$

is complete and solenoidal when $\boldsymbol{\phi}$ *ranges over the solutions of equation* (16) *that are* $m+1+\alpha$ *Hölder continuous in* Δ *and* **T** *is restricted as in Theorem* 4.

To obtain Gurtin's original forms, we set $\nu = 0$, $2\mu\mathbf{u} = \boldsymbol{\phi}$ in equation (13) and place $\boldsymbol{\phi}$ so defined into equation (17):

$$\mathbf{S} = 2\nabla(\phi+\mathbf{r}\cdot\boldsymbol{\psi})\nabla-4(\nabla\boldsymbol{\psi}+\boldsymbol{\psi}\nabla)+\nabla\times\mathbf{T}\times\nabla. \tag{18}$$

Taking $\mathbf{U} = (\phi+\mathbf{r}\cdot\boldsymbol{\psi})\mathbf{I}$ in equation (6) and replacing the repeated gradient in equation (18) by the resulting identity, equation (18) becomes

$$\mathbf{S} = -4(\nabla\boldsymbol{\psi}+\boldsymbol{\psi}\nabla)+4\mathbf{I}(\nabla\cdot\boldsymbol{\psi})+\nabla\times[\mathbf{T}+(\phi+\mathbf{r}\cdot\boldsymbol{\psi})\mathbf{I}]\times\nabla$$

which is the representation of equation (7).

On the other hand, setting $\nu = 0$, $2\mu\mathbf{g} = \boldsymbol{\phi}$ in equation (14) and substituting into equation (17) there results

$$\mathbf{S} = \nabla^2(\nabla\mathbf{g}+\mathbf{g}\nabla) - \nabla(\nabla\cdot\mathbf{g})\nabla+\nabla\times\mathbf{T}\times\nabla$$

which is equation (4).

Finally we note that the representation of equation (15) yields equation (7) and consequently nothing new.

If we employ equation (9) to describe symmetric dyadics, then on insisting **S** so expressed be solenoidal, we obtain

$$\nabla^2\mathbf{s}+\nabla\nabla\cdot\mathbf{s}-\nabla\cdot(\mathbf{U}_c\times\nabla) = 0 \tag{19}$$

Since the homogeneous part of the solution **s** leads to the same results as in the previous discussion, it suffices to consider a particular integral of equation (19). To this end, we recall Kelvin's solution, [13], of the non-homogeneous form of equation (12):

$$\mathbf{s} = -(\tfrac{1}{8}\pi)\int_\Delta [(\nabla^2 R)\mathbf{I}-(\tfrac{1}{2})\nabla\nabla R]\cdot[\nabla\cdot(\mathbf{U}_c\times\nabla)]\,\mathrm{d}Q = \nabla\times\mathbf{g}^*+\text{solution of equation (16)}$$

where $\mathbf{g}^* = -\mathcal{N}[\nabla\cdot\mathbf{U}_c]$. Taking, for example, the solutions of equation (16) in the form of equation (14), we arrive at

$$\mathbf{S} = \nabla^2(\nabla\mathbf{g}+\mathbf{g}\nabla)-\nabla(\nabla\cdot\mathbf{g})\nabla+\nabla(\nabla\times\mathbf{g}^*)+(\nabla\times\mathbf{g}^*)\nabla+\nabla\times\mathbf{U}-\mathbf{U}_c\times\nabla,$$

and by employing other forms of the solution of the homogeneous equation, we arrive at results analogous to those presented.

STRUCTURE OF S IN PERIPHRACTIC REGIONS

The additive harmonic or biharmonic vectors in equations (4), (7) or (17) constitute more generality than is necessary. Indeed, suppose that ∂D consists of the union of $n+1$ disjoint, but individually connected surfaces, $\partial D_0, \partial D_1, \ldots, \partial D_n$ such that $\partial D_1, \partial D_2, \ldots, \partial D_n$ are contained in the interior of ∂D_0. If $\mathbf{t}_i^*, \mathbf{m}_i^*(\mathbf{r}_i)$ is the resultant traction on ∂D_i $(i = 1, \ldots, n)$

where the moment $\mathbf{m}_i^*(\mathbf{r}_i)$ is computed with respect to the point $\mathbf{r}_i$ which is taken inside ∂D_i, we can express $\mathbf{S}$ in much the same way that the Kolosoff potentials [14] are treated in the case of unbalanced tractions on cavities in multiply-connected regions of the plane.

Since an important use of these forms occurs in the treatment of problems relative to isotropic, linear and homogeneous elasticity, we consider the n displacement fields

$$16\pi\mu\mathbf{u}_i(P) = -4\mathbf{t}_i^*\kappa_i + [1/(1-\nu)]\nabla\nabla\cdot[\mathbf{t}_i^* R_i] + 2\nabla\times[\mathbf{m}_i^*\kappa_i]$$

where R_i is the distance between the field point P and $\mathbf{r}_i$ and $R_i\kappa_i = 1$. If a body with elasticities λ, μ occupies the region Δ, then the corresponding stress field due to $\mathbf{u}_i$ is given by

$$\begin{aligned}16\pi\mathbf{S}_i = &-\mathbf{I}[4\nu/(1-\nu)]\nabla\cdot[\mathbf{t}_i^*\kappa_i] - 4[\nabla(\mathbf{t}_i^*\kappa_i) + (\mathbf{t}_i^*\kappa_i)\nabla] \\ &+ [2/(1-\nu)]\nabla[\nabla\cdot(\mathbf{t}_i^* R_i)]\nabla + 2\nabla[\nabla\times(\mathbf{m}_i^*\kappa_i)] + 2[\nabla\times(\mathbf{m}_i^*\kappa_i)]\nabla.\end{aligned} \tag{20}$$

For such fields, the resultant traction on any closed surface containing only the element ∂D_i of ∂D in its interior is $\mathbf{t}_i^*, \mathbf{m}_i^*(\mathbf{r}_i)$. Consequently, the stress field $\mathbf{S} - \sum_{i=1}^{n} \mathbf{S}_i$ is such that $\mathbf{t}^* = \mathbf{m}^* = 0$ for every ∂D^* in Δ. According to either [4] or [5], then

$$\mathbf{S} = \sum_{i=1}^{n} \mathbf{S}_i + \nabla\times\mathbf{T}\times\nabla \tag{21}$$

is complete when the $\mathbf{S}_i$ are defined by equation (20) and $\mathbf{T}$ meets the requirements of Theorem 4. In problems relating to continuous distributions of dislocations, it is only necessary to consider equation (1). For such circumstances, equation (20) can be simplified by setting $\nu = \infty$.

COMPATIBLE S

If $\mathbf{S}$ of equation (1) is to represent a state of stress in an equilibrated, linear, homogeneous and isotropic body, then the stress field must also satisfy

$$(1+\nu)\nabla^2\mathbf{S} + \nabla p\nabla = 0, \qquad p = \mathbf{S}:\mathbf{I} \tag{22}$$

in D. V. Blokh [15] provided $\mathbf{T}$ of equation (2) in the form

$$\mathbf{T} = (1-\nu)\nabla^2\mathbf{B} + \mathbf{I}(\nabla\cdot\mathbf{B}\cdot\nabla), \qquad \mathbf{B} = \mathbf{B}_c, \qquad \nabla^4\mathbf{B} = 0 \tag{23}$$

so as to make $\mathbf{S}$ compatible. M. Stippes [16] offered specific forms of $\mathbf{T}$ in equations (4) and (7) in terms of $\mathbf{g}$ or $\boldsymbol{\phi}$ so that the indicated stress field is compatible.

If we work, instead, with equation (21), then $\mathbf{T}$ taken as in equation (23) yields an $\mathbf{S}$ which meets equation (22). In some sense $\mathbf{T}$ of equation (23) is analogous to the Galerkin vector solution of equation (12). To continue this analogy we now derive two additional representations for $\mathbf{T}$: one similar to the Papkovich–Neuber and the second to the Naghdi–Hsu solution of the displacement equations of equilibrium.

As a result of equation (21), it is clearly sufficient to work with equation (2). For an $\mathbf{S}$ so represented, we observe from equation (6) that

$$p = \nabla\cdot\mathbf{T}\cdot\nabla - \nabla^2 t, \qquad t = \mathbf{T}:\mathbf{I}. \tag{24}$$

Since p is harmonic, equation (22) can be rewritten in the form

$$(1+\nu)\nabla^2\mathbf{T} + p\mathbf{I} = \nabla\mathbf{v} + \mathbf{v}\nabla \tag{25}$$

where $\mathbf{v}$ is any smooth vector. (Equation (25) is equation (2.3) of [15].) To solve the homogeneous form of this equation, we note that there is a biharmonic scalar b such that $p = \nabla^2 b$. As a result

$$(1+\nu)\mathbf{T}+b\mathbf{I} = (1+\nu)\mathbf{H}, \qquad \mathbf{H} = \mathbf{H}_r, \qquad \nabla^2\mathbf{H} = 0. \tag{26}$$

Computing p from equations (26) and (24), it follows that

$$(1-\nu)\nabla^2 b = -\nabla\cdot\mathbf{H}\cdot\nabla(1+\nu); \tag{27}$$

the general solution of which can be taken as

$$[(1-\nu)/(1+\nu)]b = -(\tfrac{1}{2})\mathbf{r}\cdot(\nabla\cdot\mathbf{H})-(1-\nu)h, \qquad \nabla^2 h = 0, \tag{28}$$

or

$$[(1-\nu)/(1+\nu)]b = -\mathcal{N}[\nabla\cdot\mathbf{H}\cdot\nabla]-(1-\nu)h^*, \qquad \nabla^2 h^* = 0. \tag{29}$$

Consequently

$$\mathbf{T} = \mathbf{H}+[\mathbf{I}/2(1-\nu)]\mathbf{r}\cdot(\nabla\cdot\mathbf{H})+h\mathbf{I} \tag{30}$$

or

$$\mathbf{T} = \mathbf{H}+\mathbf{I}h^*+[\mathbf{I}/(1-\nu)]N[\nabla\cdot\mathbf{H}\cdot\nabla] = \mathbf{H}^*+[\mathbf{I}/(1-\nu)]\mathcal{N}[\nabla\cdot\mathbf{H}^*\cdot\nabla] \tag{31}$$

where $\mathbf{H}^* = \mathbf{H}+\mathbf{I}h^*$. To these we could add, as noted in [15], a particular integral of equation (25) of the form $\nabla\mathbf{w}+\mathbf{w}\nabla$ where $\mathbf{w} = [1/(1+\nu)]N[\mathbf{v}]+\mathbf{h}$ where $\mathbf{h}$ is harmonic. This, however, contributes nothing to $\mathbf{S}$.

Equations (30) and (31) may be regarded as the analogues of equations (13) and (15) respectively. Blokh's solution can be obtained from equation (30) in the following way. Set $(1-\nu)\nabla^2\mathbf{B}^* = \mathbf{H}$ and take the repeated divergence of this relation. From this operation, one obtains $2(1-\nu)\nabla\cdot\mathbf{B}^*\cdot\nabla = \mathbf{r}\cdot(\nabla\cdot\mathbf{H})-[2(1-\nu)/3]h^*$ where h^* is harmonic. Calling $3\mathbf{H}^* = (h+h^*)\mathbf{I}$, $\mathbf{B}^{**} = N[\mathbf{H}^*]$ and $\mathbf{B} = \mathbf{B}^*+\beta\mathbf{B}^{**}$ where β is a suitably chosen constant, $\mathbf{T}$ of equation (30) assumes the form of equation (23).

Maxwell's stress functions

By taking $\mathbf{T} = A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk}$ in a rectangular system, it follows that equation (1) is satisfied, and choosing $\mathbf{H}$ in equation (30) as $H_1\mathbf{ii}+H_2\mathbf{jj}+H_3\mathbf{kk}$ where $\nabla^2 H_i = 0$, one arrives at a representation for Maxwell's stress functions; A, B, C which gives rise to compatible stresses. G. Morera [17] had deduced this form with $h = 0$ by a different argument.

Morera's stress functions

Setting $\mathbf{T} = N(\mathbf{ij}+\mathbf{ji})+M(\mathbf{ik}+\mathbf{ki})+L(\mathbf{jk}+\mathbf{kj})$, one again finds equation (1) to be satisfied. However, if one desires compatible stresses, it follows from equation (25) that if $\mathbf{T}$ does not contain diagonal elements, then we must include the particular integral of the previous paragraph. Calling $\mathbf{T}^*$ any one of the solutions of the homogeneous form of equation (25), then p depends only on $\mathbf{T}^*$ and consequently $\mathbf{v} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$ in equation (25) must be such that

$$p = 2\partial u/\partial x = 2\partial v/\partial y = 2\partial w/\partial z. \tag{32}$$

Furthermore, if $\mathbf{w} = u'\mathbf{i}+v'\mathbf{j}+w'\mathbf{k}$, then

$$T^*_{11} = -2\partial u'/\partial x, \qquad T^*_{22} = -2\partial v'/\partial y, \qquad T^*_{33} = -2\partial w'/\partial z. \tag{33}$$

The solution of equations (33) and (32) along with $(1+\nu)\mathbf{w} = N[\mathbf{v}]+\mathbf{h}$ yields a representation for Morera's stress functions.

REFERENCES

[1] C. E. Weatherburn, *Adv. Vector Analysis.* Bell (1928).
[2] E. Beltrami, *R. Accad. Lin. Rend.* Ser. 5, **1** (1892).
[3] G. Reider, *Abh. Braunschn. wiss. Ges.* **7**, (1960).
[4] M. E. Gurtin, *Archs ration. Mech. Analysis* **13**, (1963).
[5] D. E. Carlson, *J. Math. Analysis Appl.* (To be published.)
[6] L. Bers, F. John and J. Schecter, *Part. Diff. Eqs.* Wiley (1960).
[7] R. D. Mindlin, *Int. J. Solids Struct.* **1**, 265 (1965).
[8] M. Stippes, *Biharmonic Representations.* (In review.)
[9] M. E. Gurtin, *Archs ration. Mech. Analysis* **9** (1962).
[10] V. J. Boussinesq, *Appl. des Pot.* Paris (1885).
[11] R. D. Mindlin, *Bull. Am. math. Soc.* **42**, (1936).
[12] P. M. Naghdi and C. S. Hsu, *J. Math. Mech.* **10** (1961).
[13] A. E. H. Love, *Math. Theo. Elast.* Dover (1944).
[14] N. I. Muschelishvili, *Basic Prob. Theo. Elast.* Noordhoff (1963).
[15] V. I. Blokh, *PMM* **14** (1950).
[16] M. Stippes, *Q. appl. Math.* (To be published.)
[17] G. Morera, *Rc. Accad. Sci. Torino*, **42**, (1906).

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Résumé—Quelques expériences générales pour des champs de tension à solenoide sont obtenus par une application de la décomposition de Stokes–Helmhotz car elle se rapporte à des radicaux divalents symmétriques. Employant cela en conjonction avec les éuqations Beltrami–Michell, les solutions connues des éuqations d'élasticité classique sont déduites.

Zusammenfassung—Einige allgemeine Ausdrücke für divergenzfreie Spannungsfelder werden aus der Anwendung der Stokes-Helmholtzschen Zerlegung eines symmetrsichen Tensors erhalten. Verwendung dieser Ausdrücke in Zusammenhang mit den Beltrami-Mitchell Gleichungen ermöglicht die Ableitung der bekannten Lösungen der klassischen Elastizitätsgleichungen.

Абстракт—Получены некоторые общие выражения для полей соленоидального напряжения при применении разложения Стокс-Хелмхольца (Stokes–Helmholtz), как оно относится к симметрическим диадикам. Применяя их в сочетании с уравнениями Бельтрами–Мичела, (Beltram–Michell) выводятся известные решения уравнений классической эластичности.